IEEE Transactions on Automatic Control, 58(2):529-534, 2013

Analysis of Decentralized Quantized Auctions on Cooperative Networks

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Abstract—**This paper considers a decentralized quantity allocation problem over networks. By employing the so-called UQ-PSP auction algorithm, distributed auctions on a two-level network are developed so as to achieve efficient resource allocations (in the sense of maximization of social welfare). In the formulation in this paper, each vertex in the connected higher-level network is regarded as a supplier for a uniquely associated lower-level network, and each lower-level network consists of a set of agents which represent buyers. Each lower-level network with its associated supplier is assumed to constitute a local UQ-PSP auction, generically denoted by A_l. The adjustment of the quantities supplied to any A_l is via a consensus-based dynamical system which exchanges quantities depending upon the limit prices of the recursive bidding processes of the local auctions in the neighborhood of A_l in the higher-level network. Such a consensus auction system is formulated as a discrete-time weighted-average consensus problem with an associated family of time-varying and asymmetric Perron matrices. Subject to continuous-valued pricing, the corresponding dynamical system converges to a global limit price which is independent of the initial data and corresponds to an efficient quantity allocation. Exponential convergence is established by using the passivity property of UQ-PSP auctions, and using the SIA (stochastic, indecomposable and aperiodic) properties of the family of Perron matrices.**

I. Introduction

In this paper we study a potential application of the *Unique-limit Quantized PSP (UQ-PSP) (auction) algorithm* [1], [2], [3] to resource allocation problems on distributed networks (see e.g., [4], [5], [6]). This work is motivated by the fact that (i) agents in communication, power grid and social networks are often intrinsically unable to access and process all the information communicated over such networks but can communicate locally, and (ii) in principle, UQ-PSP based local auctions can be implemented with minimal assumptions and these have the well known desirable property of truth telling (see Sec. II-B).

Cooperative distributed decision-making via consensus algorithms has been extensively studied in computer science [7], decision theory [8], and systems and control theory [9], [10], [11]. In the present paper, we introduce two-level network structures where vertices in a higher-level network (with an arbitrary topology) are regarded as cooperative suppliers and vertices in lower-level (clique) networks are considered as noncooperative buyers, where each lower-level network is uniquely associated with a vertex in the higher-level network. In this framework, a so-called *consensus UQ-PSP auction algorithm* is formulated where the dynamics occur in two nested iterations: (i) each agent participates in a local UQ-PSP auction for a given quantity of resource; (ii) the quantities associated with local auctions are recursively adjusted via a consensus algorithm; and the cycle is then repeated until convergence occurs up to a tolerance level.

Realistic examples of cooperative suppliers as considered in this context are provided by a global network of wireless service providers and by local agents bidding for their services, and independently owned subsidiary power grids on a national network. The key motivation for the cooperation assumption in such cases would lie in the increase of social welfare of the whole network (market) which would coincide with the efficiency of the PSP and UQ-PSP auction algorithms (see [12], [3]).

This work was supported by NSERC Discovery and Strategic Grants.

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It is then assumed that viewed from the higher-level the pricing quantization is sufficiently fine that the consensus dynamics may be adequately represented by a model where the prices take values in the real number continuum $\mathcal{R}$. It will then be shown that distributed network based consensus UQ-PSP auction algorithms converge to a global limit price (which corresponds to the global market clearing price), where, moreover, global efficiency is achieved. The higher-level network consensus dynamics are formulated and analyzed as a discrete-time weighted-average consensus system with a family of time-varying and asymmetric *Perron matrices* (see (15) in Sec. III-B).

In this paper, Perron matrices [13] originate from the cooperative quantity exchange between suppliers in the higher-level network. Each supplier sends quantities to its neighbors with lower local limit prices, and receives quantities from the neighbors with higher local limit prices, while these local limit prices are updated via local auctions in the lower-level networks subject to the exchanged time-varying local quantities. The specified Perron matrices in this paper are of the form $M(k) = I - \alpha\beta(k)L$, where $\alpha > 0$ is a design parameter, $\beta(\cdot)$ is a diagonal matrix with strictly positive diagonal entries and is a system parameter of local auctions, and L is the higher-level network Laplacian (i.e., a matrix representation of the associated higher-level graph). We shall show that (i) uniform boundedness of the weightings $\beta(k)$ (i.e., the bounded relations between local quantities and local limit prices) and (ii) the SIA (stochastic, indecomposable, aperiodic) features of the Perron matrices $M(k)$, are sufficient to yield the general (geometric convergence rate) consensus properties enunciated in Theorem 3.2.

The main contributions of this paper are (i) the formulation of a two-level network based allocation system with higher-level (co-operative) consensus dynamics and lower-level (competitive) auction dynamics, and (ii) the proof of the exponential convergence of a class of consensus algorithms subject to a passivity condition on the consensus dynamics.

II. Quantized PSP Auction Algorithms

In this section we will briefly review the previous work in the Progressive Second Price (PSP) auction and the Unique-limit Quantized PSP (UQ-PSP) algorithm. The convergence results in this part will be immediately applied in Sec. III as a starting point.

A. PSP Auctions [12]

In a non-cooperative game, N agents compete for a divisible good of quantity C. Each agent $i, 1 \le i \le N$, makes a two-dimensional *bid* $s_i = (p_i, q_i)$ to a seller, where q_i is the quantity the agent desires and p_i is the unit-price the agent would like to pay for q_i. The *bidding profile* is defined as $s := [s_i]_{1 \le i \le N}$, and $s_{-i} := s \backslash \{s_i\}$ as the profile of Agent i's opponents. It is noted here that s is public information and is received by all agents in the system.

All agents in such a game update their bids by observing the present bidding profile s. The updated bidding profile is then announced publicly and agents make new bids iteratively. The bidding process repeats until some kind of equilibrium is achieved. Here we call such a bidding process a *progressive (dynamical) auction*.

Each agent is associated with a private strictly monotonic *demand function* $\delta_i : \mathcal{R}^+ \to \mathcal{R}^+, 1 \le i \le N$, whose value $\delta_i(q)$ is interpreted to be the marginal price of a certain amount of resource q for an agent i. Define $D_i = \delta_i^{-1}$, as the inverse demand function and $Z_i(q) = \int_0^q \delta_i(z)dz$, as the valuation (or reward) functions. It is assumed that $D_i(p) \ge 0$ for all $p \ge 0$.

Assumption 2.1: For $i = 1, 2, \cdots, N$, δ_i satisfies the elasticity assumption ([12], [14]), that is to say, (i) $Z_i(0) = 0$; (ii) $\delta_i \ge 0$ is decreasing and continuous; (iii) for any $C \geq q > q' \geq 0$, there exists $\gamma > 0$ such that $\delta_i(q) > 0$ implies $\delta_i(q) - \delta_i(q') < -\gamma(q - q')$. ■

Remark 2.2: The third condition of Assumption 2.1 is also referred to as a strongly monotone mapping property over $[0, C]$ for $-\delta_i$ (see [15]), that is to say, for each $-\delta_i : \mathcal{R} \to \mathcal{R}$, $(q - q')(-\delta_i(q) + \delta_i(q')) \geq \|q - q'\|^2$, for all $p, p' \in [0, C]$.

In general, the *utility* of Agent i's is defined to be

$$u_i(s) = Z_i(a_i(s)) - c_i(s), \tag{1}$$

where a_i denotes the quantity Agent i obtains by a bid s_i (when the opponents bid s_{-i}) and the charge to Agent i by the seller is denoted c_i. Since Z_i is a private valuation function, each agent's utility u_i is unknown to the other agents and the auctioneer.

The PSP *allocation rule* presented in [16] is formulated as follows:

$$a_i(s) = a_i(s_i, s_{-i}) = \min\left\{q_i, \frac{q_i}{\sum_{k:p_k=p_i} q_k} Q_i(p_i, s_{-i})\right\}, \tag{2}$$

$$c_i(s) = c_i(s_i, s_{-i}) = \sum_{j\neq i} p_j \cdot \left(a_j(0; s_{-i}) - a_j(s_i; s_{-i})\right), \tag{3}$$

where $Q_i(p, s_{-i}) = \left[C - \sum_{p_k>p,k\neq i} q_k\right]^+$, is defined the maximum available quantity at a bid price of p given s_{-i}. Here a_i corresponds to the minimum of Agent i's bid quantity q_i and the available market quantity at the bid price p_i. For the PSP allocation rule [12], the charge c_i is defined as the *(opportunity) cost* shown in (3), which is based upon the *exclusion compensation principle* [12] and the Vickrey-Clarke-Groves (VCG) mechanism [17]; it represents the potential difference in revenue between that contributed by all the other agents distinct from Agent i when (i) Agent i is absent from the auction and (ii) Agent i participates in the auction.

B. Quantized Algorithm with a Unique Iteration Limit

In this paper we assume that all agents are myopic and egoistic, that is to say, they maximize only their own utilities by reacting to the current bidding profile without considering the previous or the anticipated market bidding behaviours.

Assumption 2.3: [3] Given a *quantized price set* B_p^0 and given a bidding profile s^{k-1} at iteration $(k-1), k \geq 1$, it is assumed that all agents synchronously apply the quantized strategies $s_i^k = (p_i^k, q_i^k), 1 \leq i \leq N$, at iteration k, where $p_i^k = T\left(\delta_i, s^{k-1}, B_p^0\right) \in B_p^0$ and $q_i^k = D_i(p_i^k)$. ■

Here the quantization operator T is defined as follows (see details in [3], [18]) .

$$T\left(v_i^k, s^{k-1}, B_p^0\right) = \begin{cases} P_i(v_i^k, s_{-i}^{k-1}) = \inf\left\{p \geq 0 : C - \sum_{p_j^{k-1}>p,j\neq i} q_j^{k-1} \geq v_i^k\right\}, \\ \qquad\qquad \text{if } v_i^k = p_{\max}; \\ \min\left\{p_j \in B_p^0; p_j > P_i(v_i^k, s_{-i}^{k-1})\right\}, \text{ otherwise.} \end{cases}$$

Subject to Assumptions 2.3, the PSP auction algorithm with the allocation rules (2-3) and the utility function (1) is called a *UQ-PSP auction algorithm*. The *aggregate inverse demand function* $I(p)$ is defined by $I(p) = \sum_{1\leq i\leq N} D_i(p)$.

Theorem 2.4: [3] Subject to Assumption 2.3 and a virtual auctioneer assumption (see [3]), the price state $\{(p_i^k, q_i^k), 1 \leq i \leq N, k \geq 0\}$ of the UQ-PSP dynamical auction converges to a unique quantized price p^* in k^* iterations, where p^* and k^* satisfy

$$p^* = \min\left\{p \in B_p^0 : I(p) \leq C\right\},$$
$$k^* \leq \left|\left\{p \in B_p^0 : I(p) > C\right\}\right| + 1, \tag{4}$$

and are independent of the initial bid profile s^0. ■

Summary of key features of UQ-PSP auction algorithms

- The assumption of quantized prices is a practically meaningful and well motivated constraint, for example, in commercial auctions the bid increment amount is usually predetermined and bounded from below and therefore the pricing is not continuous, see e.g., [19], [20].
- The UQ-PSP strategy is *truthful* (w.r.t. demand functions, see [3], [12]) and is a *γ-best strategy* (w.r.t. all feasible strategies without, necessarily, the quantized pricing constraint).
- The limit states specified in Theorem 2.4 are *γ-Nash Equilibria* (i.e., no agent can gain more than $O(\gamma)$ in utility by unilaterally deviating from its strategy).
- Subject to mild assumptions on the demand functions (see [12], [21]), the limit allocation is $\sqrt{\gamma k}$ *efficient* (i.e., the social welfare is maximized modulo $\sqrt{\gamma k}$) with a constant k.
- The convergence time k^* of a UQ-PSP algorithm is only determined by the cardinality of the quantized bid price set, and is independent of the initial state and the number of agents.
- There is a tradeoff between quantization and efficiency.
- The rapid convergence and the approximate efficiency of the UQ-PSP algorithm can be extended to auctions with elastic supplies and to double auctions (see [3], [22]).

III. Resource Allocation over Networks

A. Network Based Auctions

A network based UQ-PSP auction algorithm is a distributed set of games defined on a two-level network: the higher-level network has an arbitrary connected topology and each vertex in that network is regarded as a supplier (or auctioneer) for a subnetwork lying in the set of lower-level networks which is associated solely with that supplier; each lower-level network consists of a set of agents associated to the vertices of a graph with the clique topology and each agent represents a buyer (with a slight abuse of notation, we use "agent" to denote "local buyer" in the following); each of the lower-level networks is considered to correspond to a UQ-PSP auction. It is assumed that competition takes place for a divisible resource of total quantity C over the whole network where each auctioneer allocates a quantity to its local agents based on the PSP allocation rule (2). The basic property of the network based auction is that at each (lower-level network time) cycle, the agents' recursive bids and their resource allocations are determined only by information available in the local auction; this includes the local quantity from the supplier and the set of the neighbors' bids in the local auction. As stated in the Introduction, this setup may be motivated by the fact that an agent in a large population network cannot obtain complete information (i.e. the bid profile and the total quantity) concerning all other agents and auctioneers, but can observe its immediate neighbors' behaviours. In the following we extend the formulation in Sec. II to such network based auctions.

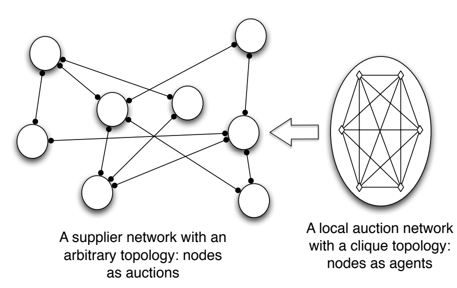

Fig. 1. Two-level network for a network based auction

The higher-level network of a two-level network system (see Fig. 1) is a connected undirected graph $G = (V, E)$ with the set of vertices $V = \{1, 2, \ldots, N\}$ (defined as suppliers), and edges $E \subseteq V \times V$ (representing the information exchange). The set of *neighbors* of each supplier $S_i, 1 \le i \le N$, is defined to be the set of vertices $N_i = \{j \in V,\ j \ne i, \{i, j\} \in E\}$. The *local network* for S_i is defined as $G_i = (V_i, E_i)$ with a clique (i.e. fully connected) topology and the *local bid profile* for the network of agents is given by $s_{G_i} = [s_{i,j}]_{1 \le j \le |V_i|}$ and $s_{(-k,G_i)} = [s_{i,j}]_{1 \le j,k \le |V_i|, j \ne k}$, where $|V_i|$ denotes the number of local agents in G_i. Agent k makes a bid by observing $s_{(-k,G_i)}$ (i.e. the neighbors of agent k consist of exactly the set of all other agents in the local network G_i), and the *local quantity* $C_i \le C$. Furthermore, in each local network G_i, Agent k, $1 \le k \le |V_i|$, receives an allocation $a_{i,k}(s_{G_i})$ and pays a cost $c_{i,k}(s_{G_i})$ under the allocation rule (2) and (3). Agent k's demand function $\delta_{i,k}$, inverse demand function $D_{i,k}$,valuation function $Z_{i,k}$ and utility function $u_{i,k}$ (i.e. the agent's preference function) are defined as before in Sec. II.

Since the information of each agent k, $1 \le k \le |V_i|$, is based only on its own local network G_i, Agent k participates in a *local UQ-PSP auction* A_i over G_i where each agent makes a two-dimensional bid in A_i at each (lower-level network) time instant. The set of local auctions $\{A_i, 1 \le i \le N\}$, together with an interaction rule between the local suppliers $S_i, 1 \le i \le N$, is called a *(two-level) network based UQ-PSP auction* or a *network based auction* in short.

B. Consensus Analysis of Local Quantities

A *global quantized limit price* p_g^* of a network based auction is defined to satisfy:

$$p_g^* = \min\{p \in B_p^0;\ \sum_{1 \le i \le N} \sum_{1 \le j \le |V_i|} D_{i,j}(p) \le C\}, \tag{5}$$

and denote $p' = \max\{p \in B_p^0; p < p_g^*\}$. The *local aggregate inverse demand function* of a local auction A_i is defined as $I_i(p) =: \sum_{1 \le j \le |V_i|} D_{i,j}(p)$.

Consider a set of local auctions $\{A_i, 1 \le i \le N\}$. If the associated set of local quantities $\{C_i, 1 \le i \le N\}$ with $\sum_i C_i = C$ satisfies, for all $1 \le i \le N$,

$$I_i(p') = \sum_{1 \le j \le |V_i|} D_{i,j}(p') > C_i \ge \sum_{1 \le j \le |V_i|} D_{i,j}(p_g^*) = I_i(p_g^*), \tag{6}$$

then it guarantees the achievement of a globally unique limit price p_g^* in each separate local UQ-PSP auction (see Theorem 2.4, and [1]). It is noted that any set of $\{C_i, 1 \le i \le N\}$ satisfying (6) corresponds to a potential efficient resource allocation subject to the quantization assumption. Furthermore, we define a *global market clearing price* $p_m, p_m \in \mathcal{R}$, which satisfies

$$\sum_{1 \le i \le N} I_i(p_m) = \sum_{1 \le i \le N} \sum_{1 \le j \le |V_i|} D_{i,j}(p_m) = C. \tag{7}$$

The continuous and strictly monotonically decreasing properties of the agents' demand functions imply (7) has a unique solution price p_m at which the summation of local quantities equals C. From (5) and (7), we immediately have $p_g^* = \min\left\{p \in B_p^0;\ p_g^* \ge p_m\right\}$.

However, for a set of fixed local quantities which do not satisfy condition (6), unequal limit prices may result in local auctions and these will reflect the gaps between the local demands and the unbalanced prior local supplies. Such unbalanced limit prices imply the inefficiency on the overall resource allocation: agents with dominant demand functions might obtain a much smaller quantity, even with much higher prices, than those with dominated demand functions. To solve this inefficiency network resource allocation problem, we formulate a (two-level) consensus UQ-PSP auction algorithm on $\{A_i, 1 \le i \le N\}$ with time-varying local quantities. As stated earlier, if local quantities can be adjusted by local suppliers based on their higher-level neighbors' market information, we can obtain the fundamental property that the global limit price p_g^* is achieved in each local UQ-PSP auction A_i corresponding to an efficient allocation to all agents.

Specification of (two-level) consensus UQ-PSP auction algorithm Let the discrete time instants $\mathcal{Z}^+ = \{1, 2, \cdots, k, \cdots\}$ be embedded in the positive real numbers $\mathcal{R}^+$. Then in the higher-level network, at each instant k, $0 \le k < \infty$, each local supplier $S_i, i \in V$, updates its quantity C_i according to the recursion

$$C_i(k+1) - C_i(k) = \sum_{j \in N_i} \Phi_{ij}(C_j(k), C_i(k), p_j^*(k), p_i^*(k)), \tag{8}$$

where $\sum_i C_i(k) = C$ for all $k \ge 0$, and hence $\sum_i \sum_j \Phi_{ij} = 0$; furthermore all *exchange functions* Φ_{ij}, which describe quantity exchange between local suppliers S_i and S_j, are assumed to be smooth, and the local limit prices $p_i^*(k) = p_i^*(C_i(k)) \ge 0$, $1 \le i \le N$, are generated by the UQ-PSP algorithm (see (4)) within each lower-level auction A_i at the interval $(k, k+1)$, given $C_i(k)$. (In the following the superscript * denotes quantization.)

In this context, we assume that all local suppliers are *cooperative* and they work together to achieve a consensus by following (8). This setup is motivated by realistic models as wireless communication networks [23] and power grids on a national network. Cooperative setup is suitable for a dynamical environment by opportunistically redistributing resources such as bandwidth, energy or real goods and consequentially improving the overall quality of service or social welfare.

Summarizing the (two-level) consensus auction algorithm into a compact form, we obtain:

Two-Level (TL) algorithm: Given an initial local quantity set $\{C_i(0); 1 \le i \le N\}$ satisfying $C_i(0) > 0$ and $\sum_i C_i(0) = C$, and an initial bid profile set $\{s_{G_i}(0)\}, 1 \le i \le N$, satisfying $p_{i,j}(0) \in B_p^0$ and $q_{i,j}(0) = D_{i,j}(p_{i,j}(0))$ for all agents $\{j, 1 \le j \le |V_i|\}$. Set $k = 0$.

(i) Given $C_i(k)$, generate a limit price $p_i^*(C_i(k))$ for each A_i via the UQ-PSP auction at iteration k;
(ii) **if** $p_i^*(C_i(k)) = p_j^*(C_j(k))$, for all pairs (i, j), $1 \le i, j \le N$, **then** stop;
(iii) **else** update the vector of the local quantities $C(k+1)$ using (8) based upon $\{p_i^*(C_i(k))\}$ and $\{C_i(k)\}$. Set $k = k+1$, and go to (i);
(iv) **end if**. ■

Since local limit prices in the consensus UQ-PSP auction algorithm implicitly indicate local demands, suppliers may update their local quantities solely based upon the limit price and quantity information from their neighbourhoods, in order to achieve a network (efficient) consensus. This is the content of Lemma 3.1 and Theorem 3.2.

Lemma 3.1: For any local auction A_i, let the quantized (valued) pricing Assumption 2.3 hold. Then the corresponding limit price function $p_i^*(C)$, given in (4), satisfies a *passivity property*:

$$(p_i^*(C_1) - p_i^*(C_2))(C_1 - C_2) \le 0, \quad \forall\ 0 < C_1, C_2 < \infty. \tag{9}$$

Sketch of Proof: Consider any local network $G_i, 1 \le i \le N$, with the associated local auction A_i. Since all agents in A_i have monotonically decreasing demand functions, the aggregate inverse demand function $I_i(p)$ is decreasing in price p. Hence (9) holds by Theorem 2.4. ■

Specification of the exchange function

Let a consensus UQ-PSP auction algorithm be defined on a connected, undirected graph $G = (V, E)$, with quantity re-allocation algorithm (8), where

$$\Phi_{ij}(C_j, C_i) = -\alpha \cdot \left(p_j^*(C_j) - p_i^*(C_i)\right), \quad \alpha > 0. \tag{10}$$

For simplicity we assume here α is known by every local supplier and is prescribed empirically at a global level. It is also feasible to associate each connection with a specified $\alpha_{ij} > 0$, all following analysis remains the same if the condition give in (16) is satisfied.

Dynamics of consensus auction systems

To study the convergence property of TL algorithm, we first formulate the collective dynamics of the consensus UQ-PSP algorithm with the exchange function (10) as follows:

$$C(k+1) = C(k) + \alpha L p^*(C(k)), \quad C(k) \in R^N,\ k \geq 0, \tag{11}$$

where $L = [l_{ij}]_{N\times N}$ is defined as the *Laplacian matrix* of a graph G [9], and $p^*(C(k)) = [p_i^*(C_i(k))] =: p^*(k)$ is uniquely determined by Theorem 2.4. Since $\mathbf{1}^T L = 0$, (11) implies $\sum_{i=1}^N C_i(k+1) = \sum_{i=1}^N C_i(k) = C$, $k \geq 0$.

Furthermore, since the elastic continuous demand assumption holds for each local demand function $\delta_{i,j}$, $1 \leq j \leq |V_i|$, $1 \leq i \leq N$, we define the *local aggregate demand function* $p_i(C_i) = I_i^{-1}(C_i)$ for each local auction A_i, which is differentiable and strictly decreasing. By definition of I_i, we have $C_i = I_i(p_i(C_i)) = \sum_{1\leq j\leq|V_i|} D_{i,j}(p_i(C_i))$. In fact, $p_i(C_i)$ is called the *local market clearing price* of the local auction A_i given C_i. For notional simplicity, we write $p_i(k) = p_i(C_i(k))$ in the following analysis. By the Mean Value Theorem, there exists a family of positive parameters $\{\beta_i(k) > 0, k \geq 0, 1 \leq i \leq N\}$ with $\beta_i(k) \in [\underline{\beta_i}, \overline{\beta_i}]$ for all $k \geq 0$ and $\underline{\beta_i} > 0$, satisfying

$$\begin{aligned} p_i(k+1) - p_i(k) &= p_i(C_i(k+1)) - p_i(C_i(k)) \\ &= -\beta_i(k)\,(C_i(k+1) - C_i(k)) \end{aligned} \tag{12}$$

Clearly we write $\beta_i(k) = \beta_i(C_i(k+1), C_i(k))$, where we note that $\beta_i(k)$ depends upon $C_i(k+1)$ and $C_i(k)$ but not directly upon k.

Overall, Equations (11) and (12) yield the dynamics of the consensus auction system as

$$p(k+1) = p(k) - \alpha\beta(k) L p^*(k), \tag{13}$$

where $\beta(k) = diag(\beta_1(k), \cdots, \beta_N(k))$ and $p(k) = p(C(k))$ is a vector with entries $p_i(k)$, $1 \leq i \leq N$.

Convergence analysis

We consider here the continuous version of the dynamical system (13) where, at each iteration k, the quantized pricing assumption is relaxed and, in particular, the quantized limit price $p_i^*(C(k)) \in B_p^0 \subset \mathcal{R}^+$ is replaced by the local market clearing price $p_i(C(k)) \in \mathcal{R}^+$. Hence (11) and (13) are replaced respectively by

$$C(k+1) = C(k) + \alpha L p(C(k)), \tag{14}$$

$$p(k+1) = p(k) - \alpha\beta(k) L p(k) = M(k) p(k), \quad k \geq 0. \tag{15}$$

where $M(k)$ is called *Perron matrix* with $M(k) := I - \alpha\beta(k) L$.

Theorem 3.2: Consider the consensus auction algorithm with local market prices $\{p_i(k), 1 \leq i \leq N, k \geq 0\}$ satisfying (14). Then for any network initial condition $[C_i(0), s_{G_i}(0)]_{1\leq i\leq N}$, there exist a unique set of limit quantities $\{C_i^\infty, 1 \leq i \leq N\}$ such that $\sum_i C_i^\infty = C$, and limit prices $p_i(C_i^\infty) = p_m$ for all $1 \leq i \leq N$, where the global market clearing price p_m is defined in (7). Moreover, for sufficiently small $\alpha > 0$ satisfying

$$0 < \alpha \max\{\beta_i(k), 1 \leq i \leq N\} < 1/\Delta, \tag{16}$$

where Δ is the maximum vertex degree in a graph G, there exist d_1, $0 < d_1 < 1$, and $c > 0$, such that $\|p(k) - \mathbf{1} p_m\| = \|p(C(k)) - \mathbf{1} p_m\| \leq c d_1^k$, $k \geq 0$, and hence the sequence $\{C(k), p(k); k \geq 0\}$, generated by (14), converges geometrically to $\{C_i^\infty, p_m; 1 \leq i \leq N\}$.

Sketch of Proof: Define $\Delta_p(k) = p(k) - \mathbf{1} p_m$. Since $L\mathbf{1}=\mathbf{0}$, (15) implies $\Delta_p(k+1) = M(k)\Delta_p(k)$. First, we can show that for all sufficiently small $\alpha > 0$, the *Perron matrix* $M(k) = I - \alpha\beta(k) L$, is (row) stochastic, indecomposable, and aperiodic (SIA) for all $k \geq 0$. (Which for any matrix A is equivalent to A stochastic and primitive, i.e., for a stochastic A there exists an integer m such that $A^m > 0$, and where both the SIA condition, and the stochastic plus primitive condition imply A has only one eigenvalue of maximum modulus 1). Second, by showing any Perron matrix product of length $(N-1)$ is scrambling [24], it is proved that a unique limit $\lim_{k\to\infty} M(k-1)M(k-2)\cdots M(0) = \mathbf{1} w^T$ of the Perron matrix product exists where $w^T \Delta_p(0) = 0$. Consequentially, $\Delta_p(k)$ is shown to converge to 0 at an exponential rate. ■

Due to space limitations, the reader is referred to [18] for the complete proof.

Subject to certain technical conditions, an error analysis (see [18]) of the difference between the continuous (lower-level prices and higher-level quantities) trajectories generated by the algorithm of equations (14) and (15) and the discrete (lower-level prices) - continuous (higher-level quantities) trajectories generated by the algorithm of equations (11) and (13) shows this difference to be uniformly bounded as the iteration count k goes to infinity.

A basic assumption for the work in this paper is that all local suppliers in the higher-level network exchange their local quantities cooperatively. Competitive supplier networks may not fit this model. Competitive suppliers can be included in the model discussed in Section II with a clique network topology, but more general situations merit analysis.

C. Numerical Simulations

Figures 2 displays a numerical simulation for a two-level network based auction with a finite quantized price set.

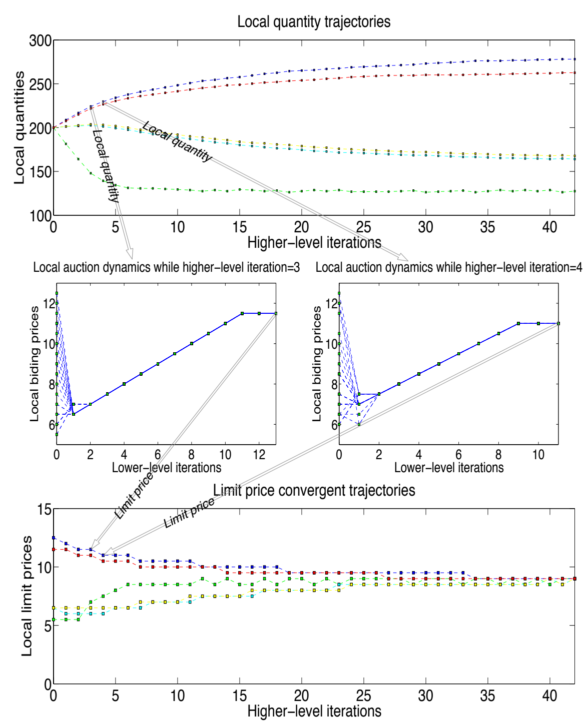

Fig. 2. A convergence process of a two-level network based auction: 5 local auctions are connected in a randomly generated connected network. Each auction has a different set of buyers (in the sense of number (respectively $|V_1| = 12, |V_2| = 16, |V_3| = 18, |V_4| = 30, |V_5| = 27$) and demand functions). The higher-level consensus is observed in 42 iterations where all local auctions agree with a unique limit quantized price while the local quantities are not necessarily equal, as is shown in the two graphs on the left. The graphs on the right depict two lower-level auction dynamics associated to one higher-level supplier during two successive higher-level iterations. It is clear that both bidding processes of the local auction converge in just 2 lower-level iterations and then converge to a steady state value (i.e., a local limit quantized price) in 11 and 9 lower-level iterations respectively.

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